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Math 12 Enriched HW Section 2.6 Inverse, Reciprocal and ABS Transformations:

1. Given that the coordinate (a,b) is on the function $y = f(x)$ , indicate what the coordinates becomes after each transformation:

a) $y = \frac{1}{f(x)}$

Reciprocal transformation $y = f(x) \to y = \frac{1}{f(x)}$

$(a,b) \to \boxed{(a, \frac{1}{b})}$

b) $y = \frac{-3}{f(4x)}$

Reciprocal transformation $y = \frac{1}{f(x)}$ $(a, \frac{1}{b})$

H.C. by $\frac{1}{4}$ $x \to 4x$ $y = \frac{1}{f(4x)}$ $(\frac{a}{4}, \frac{1}{b})$

V.E. by 3 V.R $y \to -\frac{1}{3}y$ $y = \frac{-3}{f(4x)}$ $\boxed{(\frac{a}{4}, \frac{-3}{b})}$

c) $x = f(y)$

Inverse transformation

$(a,b) \to \boxed{(b,a)}$

d) $x = \frac{-0.5}{f(3y)}$

① Reciprocal $y = \frac{1}{f(x)}$

② Inverse $x = \frac{1}{f(y)}$

③ H.C. by $\frac{1}{2}$ $x = \frac{0.5}{f(y)}$

④ H.R. $x = \frac{-0.5}{f(y)}$

⑤ V.C. by $\frac{1}{3}$ $x = \frac{-0.5}{f(3y)}$

$(a,b) \to (\frac{1}{b}, a) \to (-\frac{1}{2b}, a)$

$\boxed{(-\frac{1}{2b}, \frac{a}{3})}$

e) $y = |f(x)|$

Abs. value transformation

$(a,b) \to (a, |b|)$

f) $y = f(|x|)$

Abs. value transformation

Q1 & Q4 are mirrored onto Q3&Q2

$(a,b) \to \boxed{a, f(|a|)}$

m) $y = \frac{1}{f(x-2)+3}$

$y = f(x)$

H.S. 2R $x \to x-2$ $y = f(x-2)$ $(a+2, b)$

V.S. 3U $y \to y-3$ $y = f(x-2)+3$ $(a+2, b+3)$

Reciprocal $y = \frac{1}{f(x-2)+3}$ $\boxed{(a+2, \frac{1}{b+3})}$

n) $x = \frac{1}{f(-y+3)+2} + 2$

H.S. 3L $x \to x+3$ $y = f(x+3)$ $(a-3, b)$

H.R. $x \to -x$ $y = f(-x+3)$ $(3-a, b)$

V.S. 2U $y \to y-2$ $y = f(-x+3)+2$ $(3-a, b+2)$

Reciprocal t. $\longrightarrow y = \frac{1}{f(-x+3)+2}$ $(3-a, \frac{1}{b+2})$

Inverse t.

H.S. 2R $x \to x-2$ $\Big\} \to x = \frac{1}{f(-y+3)+2} + 2$ $(\frac{1}{b+2}+2, 3-a)$

2. Given the table of values for the function $y = f(x)$ , fill in the TOV for the other functions:

$y = f(x)$

| $x$ | $y$ |
|---|---|
| −1 | −2 |
| 0 | −1 |
| 1 | 0 |
| 2 | 1 |
| 3 | 2 |
| 5 | 4 |
| 6 | 5 |

a) $y = |f(x)|$

| $x$ | $y$ |
|---|---|
| −1 | 2 |
| 0 | 1 |
| 1 | 0 |
| 2 | 1 |
| 3 | 2 |
| 5 | 4 |
| 6 | 5 |

b) $|y| = f(x)$

| $x$ | $y$ |
|---|---|
| 3 | −2 |
| 2 | −1 |
| 1 | 0 |
| 2 | ±1 |
| 3 | 2 |
| 5 | 4 |
| 6 | 5 |

Q1 & Q2 get mirrored onto Q3 & Q4.

c) $y = f(|x|)$

| $x$ | $y$ |
|---|---|
| −1 | 0 |
| 0 | −1 |
| 1 | 0 |
| 2 | 1 |
| 3 | 2 |
| 5 | 4 |
| 6 | 5 |

Q1 & Q4 get mirrored onto Q2 & Q3

3. The point (6,12) is on $y=f(x)$. What does point become for each function below?

a) $y=3f(x-2)+4$

H.S. 2R $x \to x-2$ $y=f(x-2)$

V.E. by 3 $y \to \frac{1}{3}y$ $y=3f(x-2)$

V.S. 4 U $y \to y-4$ $y=3f(x-2)+4$

$(6,12) \to (8,12) \to (8,36) \to \boxed{(8,40)}$

b) $y=\dfrac{1}{f(x+3)}+8$ $(6,12) \to (3,12) \to (3,\frac{1}{12}) \to \boxed{(3,\frac{97}{12})}$

Reciprocal t. $y=f(x) \to y=\frac{1}{f(x)}$ $y=\frac{1}{f(x)}$

H.S. 3L $x \to x+3$ $y=\frac{1}{f(x+3)}$

V.S. 8U $y \to y-8$ $y=\frac{1}{f(x+3)}+8$

c) $y=\left|\dfrac{1}{f(2x)}\right|+7$

H.C. by $\frac{1}{2}$ $\Rightarrow x \to 2x \Rightarrow y=f(2x)$ $(3,12)$

Reciprocal t. $\to y=\frac{1}{f(2x)}$ $(3,\frac{1}{12})$

Abs. value

V.S. 7U $\Rightarrow y \to y-7 \Rightarrow y=\left|\frac{1}{f(2x)}\right|+7$ $\boxed{(3,\frac{85}{12})}$

d) $y=f^{-1}(x)$ $(6,12)$

Inverse function $y=f^{-1}(x)$ $\boxed{(12,6)}$

e) $2x=f(3y)$

Inverse reflection $x=f(y)$

H.C. by $\frac{1}{2}$ $x \to 2x$ $2x=f(y)$

V.C. by $\frac{1}{3}$ $y \to 3y$ $2x=f(3y)$

$(6,12) \to (12,6) \to \boxed{(6,2)}$

f) $y=f^{-1}(2x-1)$

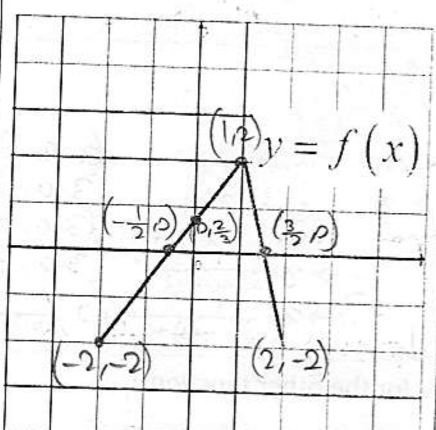
Inverse function $y=f^{-1}(x)$

H.S. 1R $x \to x-1$ $y=f^{-1}(x-1)$

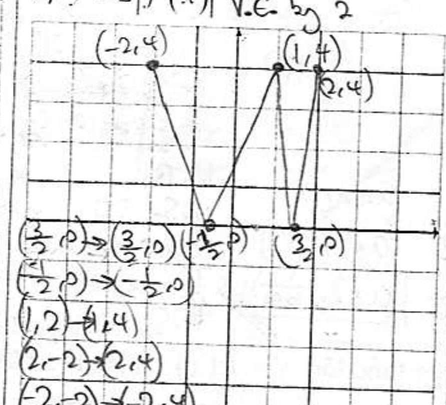
H.C. by $\frac{1}{2}$ $x \to 2x$ $y=f^{-1}(2x-1)$

$(6,12) \to (12,6) \to (13,6) \to \boxed{(\frac{13}{2},6)}$

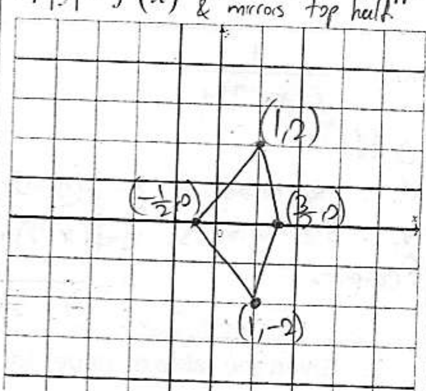
4. Given the graph of $y=f(x)$, draw the graph of the following functions:




a) $y=2|f(x)|$ Abs. value V.E. by 2

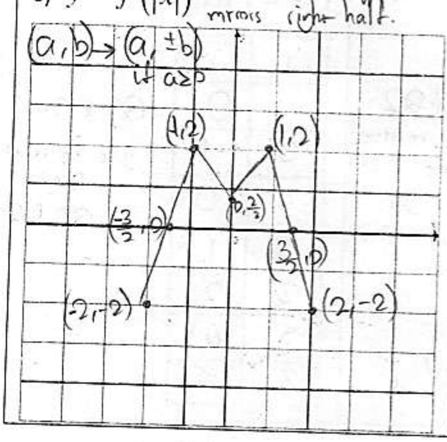



b) $|y|=f(x)$ Bottom half disappears & mirrors top half

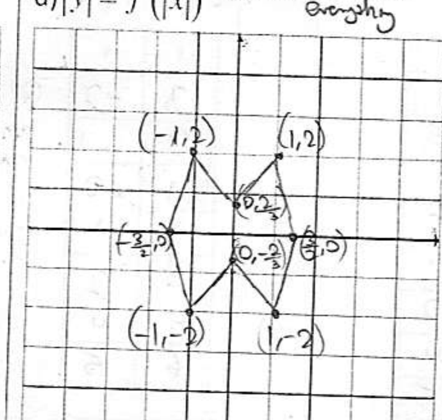



c) $y=f(|x|)$ Left half disappears and mirrors right half.

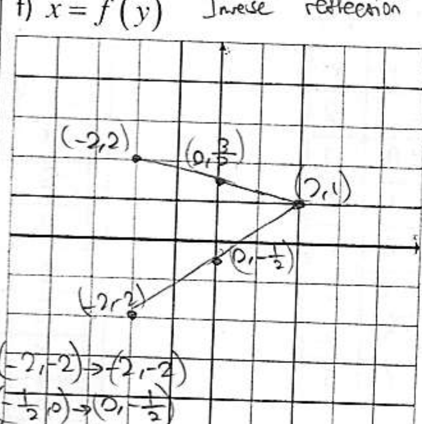
$(a,b) \to (a,\pm b)$ if $a \geq 0$




d) $|y|=f(|x|)$ Q1 mirrored onto everything




f) $x=f(y)$ Inverse reflection




$(-2,-2) \to (-2,-2)$

$(-\frac{1}{2},0) \to (0,-\frac{1}{2})$

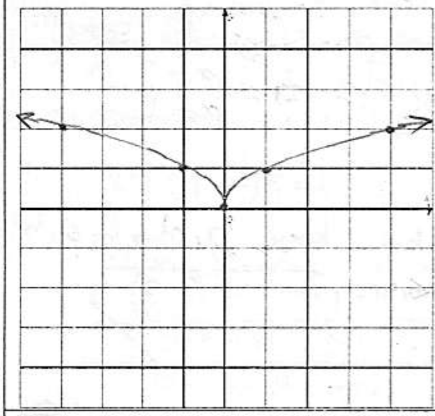
$(1,2) \to (2,1)$

$(\frac{3}{2},0) \to (0,\frac{3}{2})$

$(2,-2) \to (-2,2)$

5. Use transformations to graph the following equations:

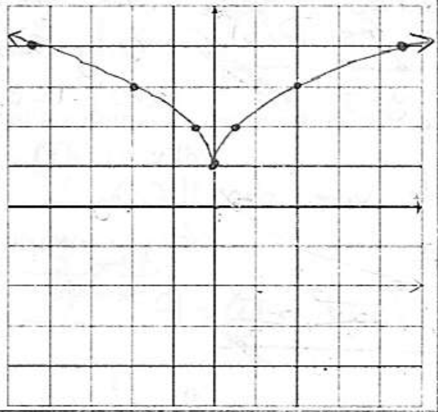
a) $y=\sqrt{|x|}$



b) $y=\sqrt{|2x|}+1$

② H.C. by $\frac{1}{2}$ ① Abs. value

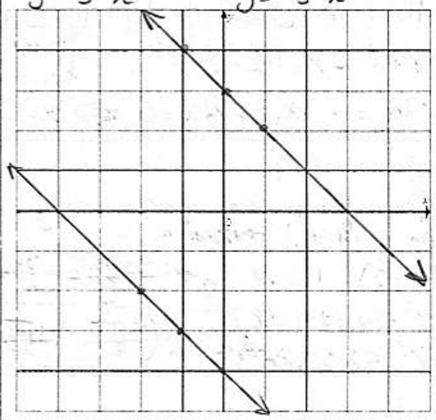
③ V.S. 1U



c) $3=|x+y|$

① $3=x+y$
$y=3-x$

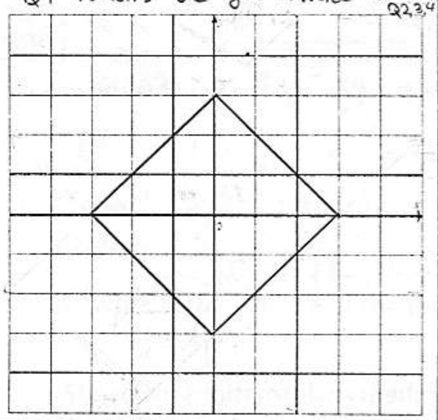
② $3=-x-y$
$y=-3-x$



d) $3=|x|+|y|$

$y=3-x \rightarrow |y|=3-|x|$

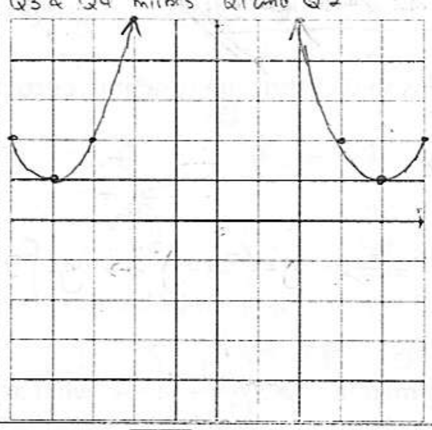
Q1 remains and gets mirrored onto Q2,3,4



e) $y=(|x|-4)^2+1$

$y=(x-4)^2+1 \rightarrow y=(|x|-4)^2+1$

Q3 & Q4 mirrors Q1 and Q2

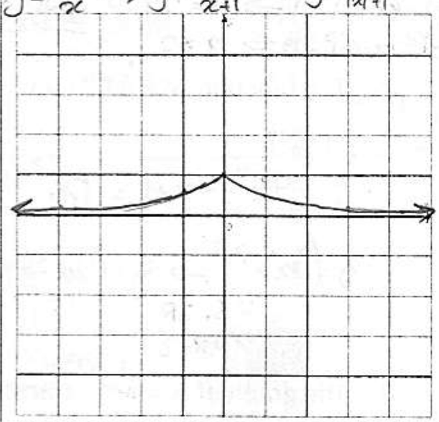


f) $y=\dfrac{1}{|x|+1}$

H.S. 1L

Abs. value (left side disappears)

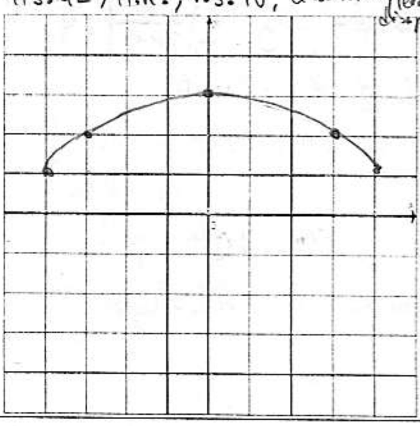
$y=\frac{1}{x} \rightarrow y=\frac{1}{x+1} \rightarrow y=\frac{1}{|x|+1}$



d) $y=\sqrt{4-|x|}+1$

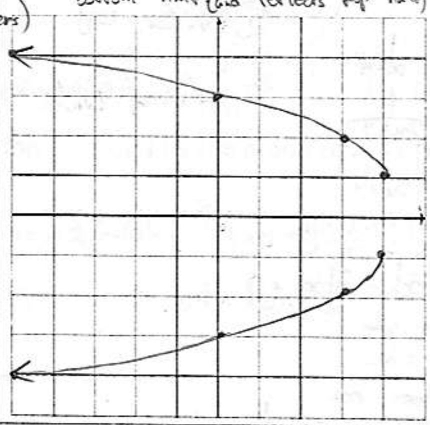
$y=\sqrt{x} \rightarrow y=\sqrt{4+x} \rightarrow y=\sqrt{4-x} \rightarrow y=\sqrt{4-|x|} \rightarrow y=\sqrt{4-|x|}+1$

HS. 4L, H.R., V.S. 1U, abs. value (left disappers)



e) $|y|=\sqrt{4-x}+1$

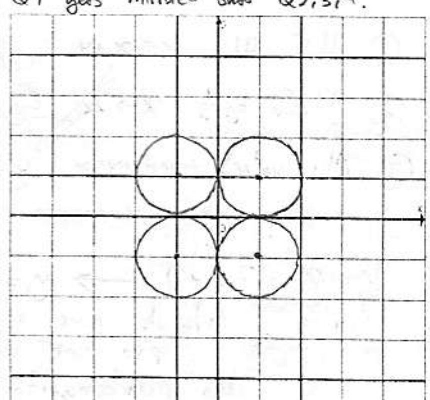
Same as (d) except with a abs. value that get rid of bottom half (and reflects top half)



f) $(|x|-1)^2+(|y|-1)^2=1$

$(x-1)^2+(y-1)^2=1 \rightarrow (|x|-1)^2+(|y|-1)^2=1$

Q1 gets mirrored onto Q2,3,4.



Circle equation: $(x-h)^2+(y-k)^2=r^2$

$(h,k)$ = centre

$r$ = radius

6. Invariant points are points that do not change after a transformation. Given the function $f(x) \Rightarrow (x-2)^2 - 1$, what are the invariant point(s) for the following transformations?

necessary as only the y-intercept stays the same after a H.R.

| | |
|---|---|
| a) $y = f(-x) = (-x-2)^2 - 1 = (x+2)^2 - 1$<br>H.R. over y-axis<br>$(x+2)^2 - \cancel{1} = (x-2)^2 - \cancel{1}$<br>$2x = -2x \Rightarrow 4x = 0 \Rightarrow x = 0 \quad y = 3$<br>only y-intercept remains the same<br>$\boxed{\text{invariant point: } (0,3)}$ | b) $y = -f(x)$<br>V.R. over x-axis,<br>only x-intercept stays the same!!!<br>$(x-2)^2 - 1 = 0 \Rightarrow x - 2 = \pm 1 \Rightarrow$ $x_1 = 3 \quad y_1 = 0$, $x_2 = 1 \quad y_2 = 0$<br>$\boxed{(3,0)}$ $\boxed{(1,0)}$ |
| c) $x = f(y)$<br>Only points on $y = x$ stay the same after an inverse reflection<br>$(x-2)^2 - 1 = x$<br>$x^2 - 4x + 4 - 1 = x$<br>$x^2 - 5x + 3 = 0$<br>$x = \frac{5 \pm \sqrt{25-12}}{2} = \frac{5 \pm \sqrt{13}}{2}$<br>$\boxed{(x,y) = \left(\frac{5+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}\right), \left(\frac{5-\sqrt{13}}{2}, \frac{5-\sqrt{13}}{2}\right)}$ | d) $y = f(4x)$<br>H.C. by $\frac{1}{4}$<br>only y-intercepts don't change $(a,b) \to \left(\frac{1}{4}a, b\right) \Rightarrow a = 0$<br>$\boxed{\text{y-intercepts = invariant points} = (0,3)}$ |
| e) $y = \frac{1}{f(x)}$<br>Reciprocal transformation; only points with y-values (1) or (−1) will remain the same. $(a,1) \to (a, \frac{1}{1})$ $(a,-1) \to (a, \frac{1}{-1})$<br>$1 = (x-2)^2 - 1 \Rightarrow x = 2 \pm \sqrt{2}$<br>$-1 = (x-2)^2 - 1 \Rightarrow x = 2$<br>$\boxed{(x,y) = (2 \pm \sqrt{2}, 1), (2,-1)}$ | f) $y = \frac{1}{2} f(x)$<br>V.C. by $\frac{1}{2}$<br>only x-intercepts remain the same:<br>$\boxed{(x,y) = (3,0), (1,0)}$ |

7. The function $y = 4x^2 + 4x + 1$ is shifted three units right to become $y = (2x - k)^2$. What is the value of "k"?

$y = 4x^2 + 4x + 1 = (2x+1)^2$

$y = (2x+1)^2 \xrightarrow[\substack{\text{H.S. 3R} \\ x \to x-3}]{} y = (2x-k)^2 \Rightarrow y = (2x+1)^2 \to y = \left[2(x-3)+1\right]^2 = (2x-5)^2$

$\therefore \boxed{k = 5}$

8. The graph of $y = \frac{1}{x}$ is transformed to $y = 3\left|\frac{1}{2x+4}\right| + 4$, what are the transformations involved? Indicate all transformations in order.

$y = \frac{1}{x} \to y = 3\left|\frac{1}{2x+4}\right| + 4$

1. H.S. 4L $\quad x \to x+4 \quad y = \frac{1}{x+4}$
2. H.C. by $\frac{1}{2}$ $\quad x \to 2x \quad y = \frac{1}{2x+4}$
3. Abs. value transformation $\quad y = \left|\frac{1}{2x+4}\right|$
4. V.E. by 3 $\quad y \to \frac{1}{3}y \quad y = 3\left|\frac{1}{2x+4}\right|$
5. V.S. 4U $\quad y \to y-4 \quad y = 3\left|\frac{1}{2x+4}\right| + 4$

(using "circle" method)

9. What are the x-intercepts of the function $y = |x|^2 - 3|x| + 2$

$y = x^2 - 3x + 2 \longrightarrow y = |x|^2 - 3|x| + 2$

Left-side gone. Right side copied onto left side.

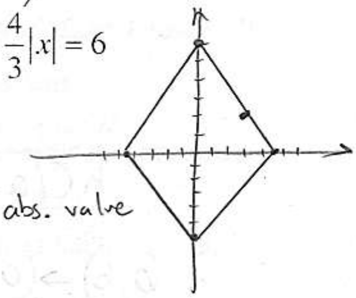
This transformation does not change the x-intercepts on the right. It just copies the x-intercepts on the right onto the left side as well.

$y = x^2 - 3x + 2 \Rightarrow y = (x-1)(x-2)$

$x_1 = 1 \quad x_2 = 2$

$\boxed{\therefore \text{x-intercepts} = (x,y) = (1,0), (2,0), (-1,0), (-2,0)}$

$y = -\frac{4}{3}x + 6$



10. What is the area encompassed by the quadrilateral created by the relation: $|y| + \frac{4}{3}|x| = 6$

Area encompassed by $y = -\frac{4}{3}x + 6$, $x$-axis, and $y$-axis:

$\frac{1}{2}\left(6 \cdot \frac{9}{2}\right) = \frac{27}{2}$

Area encompassed by $|y| + \frac{4}{3}|x| = 6$ is 4 times bigger as type IV abs. value transformation copies Q1 onto all other quadrants.

Area $= 4\left(\frac{27}{2}\right) = \boxed{54}$

11. Given that the area of the shape encompassed by the relation is 50units$^2$, then what is the value of "k"? $|y| + \frac{4}{3}|x| = k$

$y + \frac{4}{3}x = k \Rightarrow y = -\frac{4}{3}x + k$

$y$-intercept $= k$

$x$-intercept $= \frac{3}{4}k$

Using same method above (q. 10), area encompassed $= 50 = 4 \cdot \frac{1}{2}\left(k \cdot \frac{3}{4}k\right) \Rightarrow \frac{3}{2}k^2 = 50 \Rightarrow \boxed{k = \frac{10\sqrt{3}}{3}}$

$k = \cancel{\frac{-10\sqrt{3}}{3}}$ rej.

REMEMBER: complete the square to make this circle equations.

12. What is the area of the region enclosed by the graph of the equation: $x^2 + y^2 = |y| + |x|$

$x^2 + y^2 = x + y \longrightarrow |x|^2 + |y|^2 = |x| + |y|$

$x^2 - x + \frac{1}{4} + y^2 - y + \frac{1}{4} = \frac{1}{4} + \frac{1}{4}$

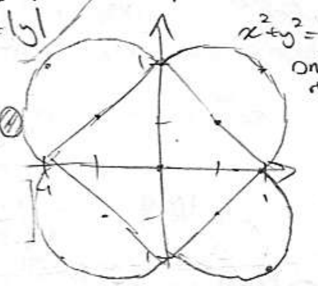
$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$

Centre $= (h, k) = \left(\frac{1}{2}, \frac{1}{2}\right)$

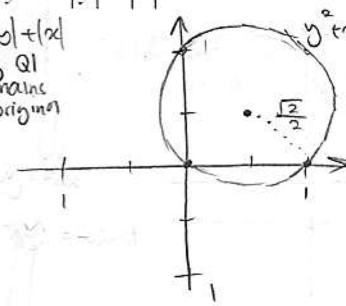
radius $= \frac{\sqrt{2}}{2} \approx 0.707$

Area $+ 2\pi$

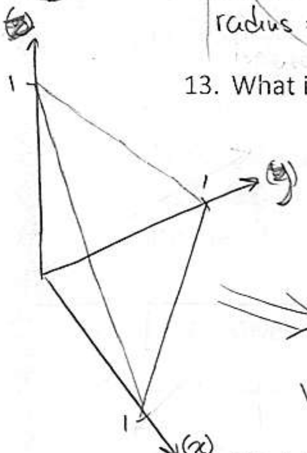
$= \boxed{2 + \pi}$









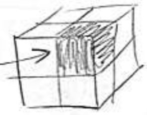




13. What is the volume of the region enclosed by the graph of $|y| + |x| + |z| \le 1$

we only care about top right corner since all other 7 sections will disappear and mirror it.

We will multiply by 8 to get all 8 sections.



$\Longrightarrow$ Volume of this tetrahedral will be $\frac{1}{8}$th of our volume.

Volume $= 8\left(\frac{1}{3} \cdot \text{base} \cdot \text{height}\right) = 8\left(\frac{1}{3} \cdot \frac{1}{2} \cdot 1\right) = \boxed{\frac{4}{3}}$

14. What is the volume of the region enclosed by the graph of $|y| + |x| + |z - 1| \le 1$

Octant = quadrant except with 8 sections.

Transformations:

① Abs. value type IV transformation $|y| + |x| + |z| \le 1$ ← octant 1 remains and octants 2,3,4,5,6,7,8 disappear and mirror octant 8.

② depth-shift 1U $z \to z-1$ $|y| + |x| + |z-1| \le 1$ ← we shift everything up by 1 on the z axis.

Our area doesn't actually change! this is b/c we shift after abs. value.

$\boxed{\text{Volume} = \frac{4}{3}}$ (according to q. 13)

15. Point $C(a, b)$ is on the graph of $y = f(x)$.

a) What point must be on the graph of $y = -\frac{1}{3}f(x+4)$?

$$y=f(x) \rightarrow y=-\frac{1}{3}f(x+4)$$

H.S. 4L, V.C. by $\frac{1}{3}$, V.R.

$$C(a,b) \rightarrow C(a-4,b) \rightarrow C\left(a-4,-\frac{b}{3}\right)$$

$\boxed{C\left(a-4,-\frac{b}{3}\right)}$

b) What point must be on the graph of $y = \frac{1}{f(x-2)} + 3$?

① Reciprocal
② H.S. 2R
③ V.S. 3U

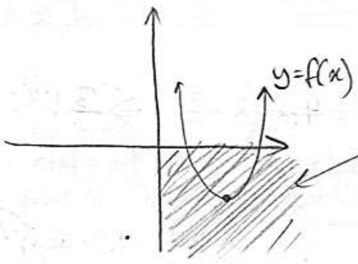
$$(a,b) \rightarrow \left(a,\frac{1}{b}\right) \rightarrow \left(a+2,\frac{1}{b}\right) \rightarrow \boxed{\left(a+2,\frac{1}{b}+3\right)}$$

c) If point $C$ is the vertex of a parabola that opens up, what is the domain and range of $y = f^{-1}(x)$? By making it an inverse function, we swap the domain and range:

$$y=f(x): \begin{array}{l} D: x \in \mathbb{R} \\ R: y \geq b \end{array} \Rightarrow \boxed{y=f^{-1}(x): \begin{array}{l} D: x \geq b \\ R: y \in \mathbb{R} \end{array}}$$

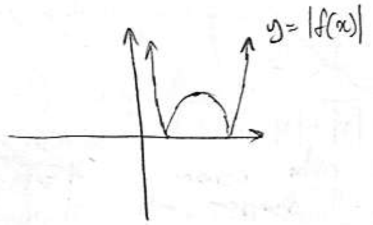
Note, $y=f^{-1}(x)$ can only be a piecewise function.

d) If point $C$ is the vertex of a parabola that opens up, and $a > 0$, $b < 0$, what is the domain and range of $y = |f(x)|$?




(a,b) is somewhere here

$\Downarrow$




$\boxed{\begin{array}{l} D: x \in \mathbb{R} \\ R: y \geq 0 \end{array}}$